
SERIES

A mathematical **series** is a fancy term for **sum**, or summation. A **finite** series might be the summation



$$1 + 2 + 5 + 6 + 10 \quad (\text{which has a **sum** of 24})$$

An **infinite** series might be something like this:

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$$

Believe it or not, this *infinite series* actually has a sum of $\frac{1}{2}$.

where the three dots (called an *ellipsis*) mean to go on and on forever (infinitely many terms being added).

Sometimes, if there's a nice pattern, a series can be written in "**sigma**" notation. The Greek capital letter sigma, Σ , is used to represent the summation (since sigma and sum both begin with the letter **s**). The following examples should explain how this notation should be read and calculated.

EXAMPLE 1: Evaluate: $\sum_{k=2}^5 (2k+1)$

Solution: The expression that determines the numbers we will add together is $2k + 1$. Below the sigma sign is the starting value of the index variable k , in this case 2; above the sigma sign is the ending value of k , in this case 5. So k starts at 2 and ends at 5, but what does k do in between? We agree that it goes up by one — in other words, k will go 2, 3, 4, and then 5. Check it out, remembering that the sigma sign, Σ , means ADD:

$$\begin{aligned}
 \sum_{k=2}^5 (2k+1) &= \overset{(k=2)}{(2 \cdot 2 + 1)} + \overset{(k=3)}{(2 \cdot 3 + 1)} + \overset{(k=4)}{(2 \cdot 4 + 1)} + \overset{(k=5)}{(2 \cdot 5 + 1)} \\
 &= 5 + 7 + 9 + 11 \\
 &= \boxed{32}
 \end{aligned}$$

EXAMPLE 2: Evaluate: $\sum_{n=0}^3 (n^2 - n)$

Solution: Does it matter that Example 1 used the index variable k and this example uses the index variable n ? Not at all — the variable used makes no difference in the final answer; it's just a placeholder. The values of n will be 0, 1, 2, and 3. For each value of n we evaluate the expression $n^2 - n$. Then we add up the results.

$$\begin{aligned}
 \sum_{n=0}^3 (n^2 - n) &= \overset{(n=0)}{(0^2 - 0)} + \overset{(n=1)}{(1^2 - 1)} + \overset{(n=2)}{(2^2 - 2)} + \overset{(n=3)}{(3^2 - 3)} \\
 &= 0 + 0 + 2 + 6 \\
 &= \boxed{8}
 \end{aligned}$$

Homework

Evaluate each series (calculate each sum):

1. $\sum_{k=3}^5 (7k - 1)$
2. $\sum_{n=2}^5 (n^2 + n)$
3. $\sum_{k=1}^6 \frac{1}{2^k}$
4. $\sum_{j=0}^4 3^j$
5. $\sum_{n=16}^{16} \frac{1}{2} \sqrt{n}$
6. $\sum_{k=3}^5 \frac{1}{k}$

$$7. \quad \sum_{m=0}^2 \frac{1}{m+1}$$

$$8. \quad \sum_{t=-1}^4 2t$$

$$9. \quad \sum_{j=-2}^2 j^3$$

For an extra challenge, write each summation in Σ -notation; use k as the index variable, and start at $k = 1$:

$$10. \quad 10 + 11 + 12 + 13$$

$$11. \quad 4^3 + 5^3 + 6^3$$

$$12. \quad 1 + \sqrt{2} + \sqrt{3} + 2$$

$$13. \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$14. \quad 1 + \sqrt[3]{2}$$

$$15. \quad \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$$

Review Problems

$$16. \quad \sum_{n=2}^5 (n^2 + n) =$$

$$17. \quad \sum_{k=1}^6 \frac{1}{2^k} =$$

$$18. \quad \sum_{j=0}^4 3^j =$$

$$19. \quad \sum_{n=16}^{16} \frac{1}{2} \sqrt{n} =$$

$$20. \quad \sum_{k=1}^{10} k =$$

$$21. \quad \sum_{n=0}^4 2^{n-2} =$$

22. True/False:

$$a. \quad \sum_{i=2}^5 i^2 = 54$$

$$b. \quad \sum_{j=0}^3 2^j = 16$$

$$c. \quad \sum_{k=-1}^1 k^3 = 3$$

$$d. \quad \sum_{L=0}^4 \sqrt{L} = 3 + \sqrt{2} + \sqrt{3}$$

23. [Only if you've studied *Combinations*]:

$$\sum_{k=0}^4 \binom{4}{k} a^{4-k} b^k =$$

This can also be written $\sum_{k=0}^4 C(4, k) a^{4-k} b^k$.

□ **TO ∞ AND BEYOND**

A. Evaluate: $\sum_{k=1}^{99} \left(\frac{1}{k} - \frac{1}{k+1} \right)$

B. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{2^k}$

C. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{k}$

D. Evaluate: $\sum_{n=1}^4 \frac{(-1)^{n+1}}{n^3}$

[Research]

[A decimal answer will be fine.]

E. Write in Σ -notation. Use k for the index variable, starting with $k = 0$.

$$\frac{1}{2} + \frac{3}{3} + \frac{5}{6} + \frac{7}{11} + \frac{9}{18} + \frac{11}{27}$$

Solutions

1. 81

2. 68

3. $\frac{63}{64}$

4. 121

5. 2

6. $\frac{47}{60}$

7. $\frac{11}{6}$

8. 18

9. 0

10. $\sum_{k=1}^4 (k+9)$

11. $\sum_{k=1}^3 (k+3)^3$

12. $\sum_{k=1}^4 \sqrt{k}$

13. $\sum_{k=1}^5 \frac{1}{k}$

14. $\sum_{k=1}^2 \sqrt[3]{k}$

15. $\sum_{k=1}^4 \frac{1}{2^{k+1}}$

16. 68

17. $\frac{63}{64}$

18. 121

19. 2

20. 55

21. $\frac{31}{4}$

22. a. T b. F c. F d. T

23. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which can also be written $(a+b)^4$

Check out: [Math is Fun - Infinite Series](#)

“The highest activity a human being can attain is learning for understanding, because to understand is to be free.”

– Baruch Spinoza